Dynamics of the anisotropic Kantowsky-Sachs geometries in R^n gravity

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We construct general anisotropic cosmological scenarios governed by an $f(R)$ gravitational sector. Focusing then on Kantowski-Sachs geometries in the case of R^n-gravity, and modelling the matter content as a perfect fluid, we perform a detailed phase-space analysis. We find that at late times the universe can result to a state of accelerating expansion, and additionally, for a particular n-range ($2 < n < 3$) it exhibits phantom behavior. Furthermore, isotropization has been achieved independently of the initial anisotropy degree, showing in a natural way why the observable universe is so homogeneous and isotropic, without relying on a cosmic no-hair theorem. Moreover, contracting solutions have also a large probability to be the late-time states of the universe. Finally, we can also obtain the realization of the cosmological bounce and turnaround, as well as of cyclic cosmology. These features indicate that anisotropic geometries in modified gravitational frameworks present radically different cosmological behaviors comparing to the simple isotropic scenarios.

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I. INTRODUCTION

The observable universe is homogeneous and isotropic at a great accuracy [1], leading the large majority of cosmological works to focus on homogeneous and isotropic geometries. The explanation of these features, along with the horizon problem, was the main reason of the inflation paradigm construction [2]. Although the last subject is well explained, the homogeneity and the isotropy problems are, strictly speaking, not fully solved, since usually one starts straightaway with a homogeneous and isotropic Friedmann-Robertson-Walker (FRW) metric, and examines the evolution of fluctuations. On the other hand, the robust approach on the subject should be to start with an arbitrary metric and show that inflation is indeed realized and that the universe evolves towards an FRW geometry, in agreement with observations. However, the complex structure of such an approach allows only for a numerical elaboration [3], and therefore in order to extract analytical solutions many authors start with one assumption more, that is investigating anisotropic but homogeneous cosmologies. This class of geometries was known since a long time ago [4], and it can exhibit very interesting cosmological behavior, either in inflationary or in post-inflationary cosmology [5]. Finally, note that such geometries may be relevant for the description of the black-hole interior [6].

The most well-studied homogeneous but anisotropic geometries are the Bianchi type (see [7, 8] and references therein) and the Kantowski-Sachs metrics [9–11], either in conventional or in higher-dimensional framework. Although some Bianchi models (for instance the Bianchi IX one) are more realistic, their complicated phase-space behavior led many authors to investigate the simpler but very interesting Bianchi I, Bianchi III, and Kantowski-Sachs geometries. In these types of metrics one can analytically examine the rich behavior and incorporate also the matter content [11–15], obtaining a very good picture of homogeneous but anisotropic cosmology.

On the other hand the observable universe is now known to be accelerating [16], and this feature led physicists to follow two directions in order to explain it. The first is to introduce the concept of dark energy (see [17] and references therein) in the right-hand-side of the field equations, which could either be the simple cosmological constant or various new exotic ingredients [18–20]. The second direction is to modify the left-hand-side of the field equations, that is to modify the gravitational theory itself, with the extended gravitational theories known as $f(R)$-gravity (see [21, 22] and references therein) being the most examined case. Such an approach can still be in the spirit of General Relativity since the only request is that the Hilbert-Einstein action should be generalized (replacing the Ricci scalar R by functions of it) asking for a gravitational interaction acting, in principle, in different ways at different scales. Such extended theories can present very interesting behavior and the corresponding cosmologies have been investigated in detail [23–27].

In the present work we are interested in investigating homogeneous but anisotropic cosmologies, focusing on Kantowski-Sachs type, in a universe governed by $f(R)$-gravity. In particular, we perform a phase-space and stability analysis of such a scenario, examining in a systematic way the possible cosmological behaviors, focusing on the late-time stable solutions. Such an approach allows us to bypass the high non-linearities of the cosmological equations, which prevent any complete analytical treatment, obtaining a (qualitative) description of the global dynamics of these models. Furthermore, in

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these asymptotic solutions we calculate various observable quantities, such are the deceleration parameter, the effective (total) equation-of-state parameter, and the various density parameters. We stress that the results of anisotropic $f(R)$ cosmology are expected to be different than the corresponding ones of $f(R)$-gravity in isotropic geometries, similarly to the differences between isotropic [28] and anisotropic [29] considerations in General Relativity. Additionally, the results are expected to be different from anisotropic General Relativity, too. As we see, anisotropic $f(R)$ cosmology can be consistent with observations.

The paper is organized as follows: In section II we construct the cosmological scenario of anisotropic $f(R)$-gravity, presenting the kinematical and dynamical variables, particularizing on the $f(R) = R^n$ ansatz in the case of Kantowski-Sachs geometry. Having extracted the cosmological equations, in section III we perform a systematic phase-space and stability analysis of the system. Thus, in section IV we analyze the physical implications of the obtained results, and we discuss the cosmological behaviors of the scenario at hand. Finally, our results are summarized in section V.

II. ANISOTROPIC $f(R)$ COSMOLOGY

In this section we present the basic features of anisotropic Bianchi I, Bianchi III, and Kantowski-Sachs geometries. After presenting the kinematical and dynamical variables in the first two subsections, we focus on the Kantowski-Sachs geometrical background and on the R^n-gravity in the last two subsections.

A. The geometry and kinematical variables

In order to investigate anisotropic cosmologies, it is usual to assume an anisotropic metric of the form [12, 30]:

$$ds^2 = -N(t)^2 dt^2 + [e_1{}^1(t)]^{-2} dr^2 + [e_2{}^2(t)]^{-2}[d\vartheta^2 + S(\vartheta)^2\, d\varphi^2], \tag{1}$$

where $1/e_1{}^1(t)$ and $1/e_2{}^2(t)$ are the expansion scale factors. The frame vectors in coordinate form are written as

$$\begin{aligned} &\mathbf{e}_0 = N^{-1}\partial_t, \quad \mathbf{e}_1 = e_1{}^1\partial_r \\ &\mathbf{e}_2 = e_2{}^2\partial_\vartheta, \quad \mathbf{e}_3 = e_2{}^2/S(\vartheta)\partial_\varphi. \end{aligned} \tag{2}$$

The metric (1) can describe three geometric families, that is

$$S(\vartheta) = \begin{cases} \sin\vartheta & \text{for} \quad k=+1, \\ \vartheta & \text{for} \quad k=0, \\ \sinh\vartheta & \text{for} \quad k=-1, \end{cases} \tag{3}$$

known respectively as Kantowski-Sachs, Bianchi I and Bianchi III models.

In the following, we will focus on the Kantowski-Sachs geometry [10], since it is the most popular anisotropic model, and since all solutions are known analytically in the case of General Relativity, even if some particular types of matter are coupled to gravity [11, 12]. These are spatially homogeneous spherically symmetric models [30, 31], with 4 Killing vectors: ∂_φ, $\cos\varphi\ \partial_\vartheta - \sin\varphi\cot\vartheta\ \partial_\varphi$, $\sin\varphi\ \partial_\vartheta + \cos\varphi\cot\vartheta\ \partial_\varphi, \partial_x$ [32].

Let us now consider relativistic fluid dynamics (see for e.g. [33]) in such a geometry. For any given fluid 4-velocity vector field u^μ, the projection tensor[1]

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$$

projects into the instantaneous rest-space of a comoving observer. It is standard to decompose the first covariant derivative $\nabla_\mu u_\nu$ into its irreducible parts

$$\nabla_\mu u_\nu = -u_\mu \dot{u}_\nu + \sigma_{\mu\nu} + \frac{1}{3}\Theta h_{\mu\nu} - \omega_{\mu\nu},$$

where $\sigma_{\mu\nu}$ is the symmetric and trace-free shear tensor ($\sigma_{\mu\nu} = \sigma_{(\mu\nu)}$, $\sigma_{\mu\nu}\, u^\nu = 0$, $\sigma^\mu{}_\mu = 0$), $\omega_{\mu\nu}$ is the antisymmetric vorticity tensor ($\omega_{\mu\nu} = \omega_{[\mu\nu]}$, $\omega_{\mu\nu}\, u^\nu = 0$) and $\dot{u}_\mu$ is the acceleration vector defined as $\dot{u}_\mu = u^\nu\nabla_\nu u_\mu$ (a dot denotes the derivative with respect to t). In the above expression we have introduced also the volume expansion scalar. In particular,

$$\Theta = \nabla_\mu u^\mu, \tag{4}$$

which defines a length scale ℓ along the flow lines, describing the volume expansion (contraction) behavior of the congruence completely, via the standard relation, namely

$$\Theta \equiv \tfrac{3\dot{\ell}}{\ell}. \tag{5}$$

In cosmological contexts it is customary to use the Hubble scalar $H = \Theta/3$ [34]. From these it becomes obvious that in FRW geometries, ℓ coincides with the scale factor. Finally, it can be shown that these kinematical fields are related through [33, 34]

$$\sigma_{\mu\nu} := \dot{u}_{(\mu}u_{\nu)} + \nabla_{(\mu}u_{\nu)} - \frac{1}{3}\Theta h_{\mu\nu} \tag{6}$$

$$\omega_{\mu\nu} := -u_{[\mu}\dot{u}_{\nu]} - \nabla_{[\mu}u_{\nu]}. \tag{7}$$

Transiting to the synchronous temporal gauge, we can set N to any positive function of t, or simply $N = 1$. Thus, we extract the following restrictions on kinematical variables:

$$\sigma^\mu_\nu = \mathrm{diag}(0, -2\sigma_+, \sigma_+, \sigma_+), \quad \omega_{\mu\nu} = 0, \tag{8}$$

where

$$\sigma_+ = \frac{1}{3}\frac{d}{dt}\left[\ln\frac{e_1{}^1}{e_2{}^2}\right]. \tag{9}$$

[1] Covariant spacetime indices are denoted by letters from the second half of the Greek alphabet.

Finally, note that the Hubble scalar can be expressed in terms of ${e_1}^1$ and ${e_2}^2$ as

$$H = -\frac{1}{3}\frac{d}{dt}\left[\ln {e_1}^1({e_2}^2)^2\right]. \tag{10}$$

B. The action and dynamical variables

Let us now construct $f(R)$ cosmology in such a geometrical background. In the metric formalism the action for $f(R)$-gravity is given by [21, 22]

$$\mathcal{S}_{\text{metric}} = \int dx^4\sqrt{-g}\left[f(R) - 2\Lambda + \mathcal{L}^{(m)}\right], \tag{11}$$

where $f(R)$ is a function of the Ricci scalar R, and $\mathcal{L}^{(m)}$ accounts for the matter content of the universe. Additionally, we use the metric signature $(-1, 1, 1, 1)$, Greek indices run from 0 to 3, and we impose the standard units in which $c = 8\pi G = 1$. Finally, in the following, and without loss of generality, we set the usual cosmological constant $\Lambda = 0$.

The fourth-order equations obtained by varying the action (11) with respect to the metric write:

$$G_{\mu\nu} = \frac{T^{(m)}_{\mu\nu}}{f'(R)} + T^R_{\mu\nu}, \tag{12}$$

where the prime denotes differentiation with respect to R. In this expression $T^{(m)}_{\mu\nu}$ denotes the matter energy-momentum tensor, which is assumed to correspond to a perfect fluid with energy density ρ_m and pressure p_m. Additionally, $T^R_{\mu\nu}$ denotes a correction term describing a "curvature-fluid" energy-momentum tensor of geometric origin [35, 36]:

$$T^R_{\mu\nu} = \frac{1}{f'(R)}\left[\frac{1}{2}g_{\mu\nu}\left(f(R) - Rf'(R)\right) + \nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R)\right], \tag{13}$$

where ∇_μ is the covariant derivative associated to the Levi-Civita connection of the metric and $\Box \equiv \nabla^\mu\nabla_\mu$. Note that in the last two terms of the right hand side there appear fourth-order metric-derivatives, justifying the name "fourth order gravity" used for this class of theories [22]. By taking the trace of equation (12) and re-ordering terms one obtains the "trace equation" (equation (5) in section IIA of [37])

$$3\Box f'(R) + Rf'(R) - 2f(R) = T, \tag{14}$$

where $T = T^\mu_\mu$ is the trace of the energy-momentum tensor of ordinary matter.

In the phenomenological fluid description of a general matter source, the standard decomposition of the energy-momentum tensor $T_{\mu\nu}$ with respect to a timelike vector field u^μ is given by

$$T_{\mu\nu} = \mu u_\mu u_\nu + 2q_{(\mu}u_{\nu)} + Ph_{\mu\nu} + \pi_{\mu\nu}, \tag{15}$$

where μ denotes the energy density scalar, P is the isotropic pressure scalar, q_μ is the energy current density vector ($q_\mu\, u^\mu = 0$) and $\pi_{\mu\nu}$ is the trace-free anisotropic pressure tensor ($\pi_{\mu\nu}u^\nu = 0$, $\pi^\mu_\mu = 0, \pi_{\mu\nu} = \pi_{\nu\mu}$).

The matter fields need to be related through an appropriate thermodynamical equation of state in order to provide a coherent picture of the physics underlying the fluid spacetime scenario. Applying this covariant decomposition to the "curvature-fluid" energy-momentum tensor (13) we obtain

$$\mu = -\frac{1}{2}\left[\frac{f(R) - Rf'(R) + 6H\frac{d}{dt}f'(R)}{f'(R)}\right]$$
$$P = -\frac{1}{2}\left[\frac{-f(R) + Rf'(R) - 4H\frac{d}{dt}f'(R) - 2\frac{d^2}{dt^2}f'(R)}{f'(R)}\right]. \tag{16}$$

Finally, the anisotropic pressure tensor is given by $\pi^\mu_\nu = \mathrm{diag}(0, -2\pi_+, \pi_+, \pi_+)$, where

$$\pi_+ = -\frac{\frac{d}{dt}f'(R)}{f'(R)}\sigma_+. \tag{17}$$

C. The cosmological equations

Let us now present the cosmological equations in the homogeneous and anisotropic Kantowski-Sachs metric. In particular, the Einstein's equations (12) in the Kantowski-Sachs metric read

$$\begin{aligned}
3\sigma_+^2 - 3H^2 - {}^2K &= -\mu - \frac{\rho_m}{f'(R)},\\
-3(\sigma_+ + H)^2 - 2\dot{\sigma_+} - 2\dot{H} - {}^2K &= \frac{p_m}{f'(R)} + P - 2\,\pi_+,\\
-3\sigma_+^2 + 3\sigma_+H - 3H^2 + \dot{\sigma_+} - 2\dot{H} &= \frac{p_m}{f'(R)} + P + \pi_+.
\end{aligned} \tag{18}$$

In these expressions ρ_m and p_m are the energy density and pressure of the matter perfect fluid, and their ratio gives the matter equation-of-state parameter

$$w = \frac{p_m}{\rho_m}. \tag{19}$$

Furthermore, ${}^2K$ is the Gauss curvature of the 3-spheres [34] given by

$${}^2K = ({e_2}^2)^2, \tag{20}$$

and its evolution equation writes

$${}^2\dot{K} = -2(\sigma_+ + H)\left({}^2K\right). \tag{21}$$

Additionally, the evolution equation for ${e_1}^1$ reads (see equation (42) in section 4.1 of [30])

$$\dot{e_1}^1 = -\left(H - 2\sigma_+\right){e_1}^1. \tag{22}$$

Finally, observing the form of the first of equations (18) one can define the various density parameters of the scenario at hand, namely [38] the curvature one

$$\Omega_k \equiv -\frac{{}^2K}{3H^2}, \tag{23}$$

the matter one

$$\Omega_m \equiv \frac{\rho_m}{3H^2 f'(R)}, \tag{24}$$

the "curvature-fluid" one

$$\Omega_{\text{curv.fl}} \equiv \frac{\mu}{3H^2}, \tag{25}$$

and the shear one

$$\Omega_\sigma \equiv \left(\frac{\sigma_+}{H}\right)^2, \tag{26}$$

satisfying $\Omega_k + \Omega_m + \Omega_{\text{curv.fl}} + \Omega_\sigma = 1$.

Now, the trace equation (14) in the Kantowski-Sachs metric reads

$$-3\frac{d^2}{dt^2}f'(R) - 9H\frac{d}{dt}f'(R) + Rf'(R) - 2f(R) = -\rho_m + 3p_m, \tag{27}$$

and the Ricci scalar writes

$$R = 12H^2 + 6\sigma_+^2 + 6\dot{H} + 2\,{}^2K. \tag{28}$$

At this stage we can reduce equations (18) along with (27), with respect to $\dot{H}$, $\dot{\sigma_+}$ and ${}^2K$ [39], acquiring the Raychaudhuri equation

$$\dot{H} = -H^2 - 2\sigma_+^2 - \frac{1}{6f'(R)}[\rho_m + 3p_m] - \frac{H}{2f'(R)}\frac{d}{dt}f'(R) + \frac{1}{6}\left[R - \frac{f(R)}{f'(R)} - \frac{3}{f'(R)}\frac{d^2}{dt^2}f'(R)\right], \tag{29}$$

the shear evolution

$$\dot{\sigma_+} = -\sigma_+^2 - 3H\sigma_+ + H^2 - \frac{\rho_m}{3f'(R)} - \frac{1}{6}\left[R - \frac{f(R)}{f'(R)}\right] + \frac{(H - \sigma_+)}{f'(R)}\frac{d}{dt}f'(R), \tag{30}$$

and the Gauss constraint

$${}^2K = 3\sigma_+^2 - 3H^2 + \frac{\rho_m}{f'(R)} + \frac{1}{2}\left[R - \frac{f(R)}{f'(R)}\right] - \frac{3H}{f'(R)}\frac{d}{dt}f'(R). \tag{31}$$

It proves convenient to use the trace equation (27) to eliminate the derivative $\frac{d^2}{dt^2}f'(R)$ in (29), obtaining a simpler form of Raychaudhuri equation, namely

$$\dot{H} = -H^2 - 2\sigma_+^2 - \frac{\rho_m}{3f'(R)} + \frac{f(R)}{6f'(R)} + \frac{1}{f'(R)}H\frac{d}{dt}f'(R). \tag{32}$$

Furthermore, the Gauss constraint can alternatively be expressed as

$$\left[H + \frac{1}{2}\frac{\frac{df'(R)}{dt}}{f'(R)}\right]^2 + \frac{1}{3}{}^2K = \sigma_+^2 + \frac{\rho_m}{3f'(R)} + \frac{1}{6}\left[R - \frac{f(R)}{f'(R)}\right] + \frac{1}{4}\left[\frac{\frac{df'(R)}{dt}}{f'(R)}\right]^2. \tag{33}$$

In summary, the cosmological equations of $f(R)$-gravity in the Kantowski-Sachs background are the "Raychaudhuri equation" (32), the shear evolution (30), the trace equation (27), the Gauss constraint (33), the evolution equation for the 2-curvature ${}^2K$ (21), and the evolution equation for ${e_1}^1$ (22). Finally, these equations should be completed by considering the evolution equations for matter sources.

D. R^n-gravity

In this subsection we specify the above general cosmological system. In particular, in order to continue we have to make an assumption concerning the function $f(R)$. We impose an ansatz of the form $f(R) = R^n$ [21, 22, 31, 40], since such an ansatz does not alter the characteristic length scale (and General Relativity is recovered when $n = 1$), and it leads to simple exact solutions which allow for comparison with observations [41, 42]. Additionally, following [35, 36] we consider that the parameter n is related to the matter equation-of-state parameter through

$$n = \frac{3}{2}(1 + w). \tag{34}$$

Such a choice is imposed by the requirement of the existence of an Einstein static universe in FRW backgrounds, which leads to a severe constraint in $f(R)$, namely $f(R) = 2\Lambda + R^{\frac{3}{2}(1+w)}$ with $w \neq -1$. Inversely, it can be seen as the equation-of-state parameter of ordinary matter is a fixed function of n, if we desire Einstein static solutions to exist. The existence of such a static solution (in practice as a saddle-unstable one) is of great importance in every cosmological theory, since it connects the Friedmann matter-dominated phase with the late-time accelerating phase, as required by observations [35, 36]. Thus, if we relax the condition $n = \frac{3}{2}(1 + w)$ in R^n-gravity, in general we cannot obtain the epoch-sequence of the universe. Finally, note that the constraint $-1/3 \leq w \leq 1$ that arises from the satisfaction of all the energy conditions for standard matter, imposes bounds on n, namely $n \in [1, 3]$, with the most interesting cases being those of dust ($w = 0$, $n = 3/2$) and radiation fluid ($w = 1/3$, $n = 2$).

The cosmological equations for R^n-gravity, in the homogeneous and anisotropic Kantowski-Sachs metric (1), are obtained by specializing equations (32), (30), (33), (27), (21), (22) and consider the conservation equation for standard matter.

Equations (32), (30), (33) and (27), become respectively

$$\dot{H}+H^2=-2\sigma_+^2-\frac{\rho_m}{3nR^{n-1}}+\frac{1}{6n}R+(n-1)H\frac{\dot{R}}{R}, \quad (35)$$

$$\dot{\sigma_+}=-\sigma_+^2-3H\sigma_++H^2-\frac{\rho_m}{3nR^{n-1}}-\frac{n-1}{6n}R -(n-1)\left(\sigma_+-H\right)\frac{\dot{R}}{R}, \quad (36)$$

$$\left(H+\frac{n-1}{2}\frac{\dot{R}}{R}\right)^2+\frac{1}{3}{}^2K=\sigma_+^2+\frac{\rho_m}{3nR^{n-1}} +\frac{n-1}{6n}R+\frac{(n-1)^2}{4}\frac{\dot{R}^2}{R^2}, \quad (37)$$

$$\frac{\ddot{R}}{R}=\frac{n-2}{3n(n-1)}R+\frac{2(2-n)}{3n(n-1)}\frac{\rho_m}{R^{n-1}}-3H\frac{\dot{R}}{R}-(n-2)\frac{\dot{R}^2}{R^2}. \quad (38)$$

Additionally, we consider the evolution equation for the 2-curvature ${}^2K$ given by (21), as well as the matter conservation equation

$$\dot{\rho}_m=-2nH\rho_m. \quad (39)$$

Finally, in order to close the equation-system we consider also the propagation equation (22).

III. PHASE-SPACE ANALYSIS

In the previous section we formulated $f(R)$-gravity in the case of the homogeneous and anisotropic Kantowski-Sachs geometry, focusing on the $f(R)=R^n$ ansatz. Having extracted the cosmological equations we can investigate the possible cosmological behaviors and discuss the corresponding physical implications by performing a phase-space analysis. Such a procedure bypasses the complexity of the cosmological equations and provides us the understanding of the dynamics of these scenarios.

A. The dynamical system

In order to perform the phase-space and stability analysis of the models at hand, we have to transform the cosmological equations into an autonomous dynamical system [43]. However, since the present system is more complicated, in order to avoid ambiguities related to the non-compactness at infinity we define compact variables that can describe both expanding and collapsing models [31, 44]. This will be achieved by introducing the auxiliary variables [45, 46]:

$$\begin{aligned}&Q=\frac{H}{D},\quad \Sigma=\frac{\sigma_+}{D},\\ &x=\frac{(n-1)\dot{R}}{2RD},\quad y=\frac{(n-1)R}{6nD^2},\quad z=\frac{\rho_m}{3nR^{n-1}D^2},\\ &K=\frac{{}^2K}{3D^2},\quad {E_1}^1=\frac{{e_1}^1}{D},\end{aligned} \quad (40)$$

where we have defined

$$D=\sqrt{\left(H+\frac{n-1}{2}\frac{\dot{R}}{R}\right)^2+\frac{1}{3}{}^2K}. \quad (41)$$

Furthermore, we introduce the time variable τ through

$$d\tau=\left(\frac{D}{n-1}\right)dt, \quad (42)$$

and from now on primes will denote derivatives with respect to τ. Finally, note that in the following we focus on the general case $n\neq 1$. The dynamical investigation of Kantowsky-Sachs model in General Relativity (that is for $n=1$) has been performed in [47].

In terms of these auxiliary variables the Gauss constraint (37) becomes

$$x^2+y+z+\Sigma^2=1. \quad (43)$$

Moreover, the D-definition (41) becomes

$$(Q+x)^2+K=1. \quad (44)$$

Finally, from the definitions (40) we obtain the bounds $y\geq 0$, $z\geq 0$ and $K\geq 0$. Therefore, we conclude that the auxiliary variables must be compact and lie at the intervals

$$\begin{aligned}&Q\in[-2,2],\quad \Sigma\in[-1,1],\\ &x\in[-1,1],\quad y\in[0,1],\quad z\in[0,1]\\ &K\in[0,1],\end{aligned} \quad (45)$$

while ${E_1}^1$ is un-constrained.

In summary, using the dimensionless auxiliary variables (40), along with the two constraints (43) and (44), we reduce the complete cosmological system to a five-dimensional one given by:

$$
\begin{aligned}
Q' = &(1-n)\Sigma Q^3 - (n-1)\left[-3x^2 + 2\Sigma x + (n-3)\Sigma^2 + n\left(x^2+y-1\right)+1\right]Q^2 \\
&-(n-1)\left\{x\left[-3x^2+(x-3\Sigma)\Sigma + n\left(x^2+\Sigma^2+y-1\right)-1\right]-\Sigma\right\}Q \\
&-x^2+\Sigma^2+n\left(x^2-\Sigma^2+y-1\right)+1,
\end{aligned} \tag{46}
$$

$$
\begin{aligned}
\Sigma' = &-(n-3)(n-1)(Q+x)\Sigma^3 - (n-1)(Q+x-1)(Q+x+1)\Sigma^2 \\
&-(n-1)(Q+x)\left[(n-3)\left(x^2-1\right)+ny\right]\Sigma + (n-1)\left[(Q+x)^2-1\right],
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
x' = &-(n-3)(n-1)x^4 - (n-1)\left[(n-3)Q+\Sigma\right]x^3 - (n-1)\left[-3\Sigma^2+2Q\Sigma+n\left(\Sigma^2+y-2\right)+5\right]x^2 \\
&-(n-1)\left\{Q\left[(Q-3\Sigma)\Sigma+n\left(\Sigma^2+y-1\right)+3\right]-\Sigma\right\}x + (n-2)\left[-\Sigma^2+n\left(\Sigma^2+y-1\right)+1\right],
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
y' = &-2(n-1)n(Q+x)y^2 + y\left\{-2\left[(n-4)n+3\right](Q+x)\Sigma^2 - 2(n-1)(Q+x-1)(Q+x+1)\Sigma\right. \\
&\left.-2\left\{(Q+x)\left(x^2-1\right)n^2+\left(-4x^3-4Qx^2+2x+Q\right)n+x[3x(Q+x)-2]\right\}\right\},
\end{aligned} \tag{49}
$$

$$
\left({E_1}^1\right)' = {E_1}^1(n-1)\left\{-(n-3)(Q+x)\Sigma^2-\left[(Q+x)^2-3\right]\Sigma+(-Q-x)\left[-3x^2+n\left(x^2+y-1\right)+1\right]\right\}. \tag{50}
$$

Proceeding forward, as we observe, equation (50) decouples from (46)-(49). Thus, if one is only interested in the dynamics of Kantowsky-Sachs $f(R)$-models, all the relevant information of the system is given by the remaining equations. Therefore, we can reduce our analysis to the system (46)-(49), defined in the compact phase-pace

$$
\Psi = \left\{x^2+y+\Sigma^2 \leq 1, |Q+x| \leq 1, x \in [-1,1],\, y \in [0,1],\, \Sigma \in [-1,1], Q \in [-2,2]\right\}. \tag{51}
$$

Finally, while for the purpose of the present work it is adequate to investigate the system (46)-(49), leaving outside the decoupled equation (50), for completeness we study equation (50) in Appendix B.

B. Invariant sets and critical points

Using the auxiliary variables (40) the cosmological equations of motion (35)-(39), were transformed into the autonomous form (46)-(50), that is in a form $\mathbf{X}' = \mathbf{f(X)}$, where $\mathbf{X}$ is the column vector constituted by the auxiliary variables and $\mathbf{f(X)}$ the corresponding column vector of the autonomous equations. The critical points $\mathbf{X_c}$ are extracted satisfying $\mathbf{X}' = \mathbf{0}$, and in order to determine the stability properties of these critical points we expand around $\mathbf{X_c}$, setting $\mathbf{X} = \mathbf{X_c} + \mathbf{U}$ with $\mathbf{U}$ the perturbations of the variables considered as a column vector. Thus, up to first order we acquire $\mathbf{U}' = \mathbf{\Xi} \cdot \mathbf{U}$, where the matrix $\mathbf{\Xi}$ contains the coefficients of the perturbation equations. Therefore, for each critical point, the eigenvalues of $\mathbf{\Xi}$ determine its type and stability. In particular, eigenvalues with negative (positive) real parts correspond to a stable (unstable) point, while eigenvalues with real parts of different sign correspond to a saddle point. Lastly, when at least one eigenvalue has zero real part, the corresponding point is a non-hyperbolic one.

Now, there are several invariant sets, that is areas of the phase-space that evolves to themselves under the dynamics, for the dynamical system (46)-(49). There are two invariant sets given by $Q+x=\pm 1$, corresponding to $K=0$, that is ${e_2}^2/D = 0$. However, the solutions of the evolution equations inside this invariant set do not correspond to exact solutions of the field equations, since the frame variables has to satisfy $\det(\mathbf{e}_a) \neq 0$. Nevertheless, it appears that this invariant set plays a fundamental role in describing the asymptotic behavior of cosmological models.

Another invariant set is $y=0$, that is $\rho_m = 0$, $R \equiv 12H^2 + 6\dot{H} + 6\sigma^2 + 2^2 K = 0$, provided D is finite, or $\rho_m = 0$, $R = R_0$, with R_0 a constant, provided $D \to \infty$. This invariant set contains vacuum (Minkowsky) and static models.

Another invariant set appears in the case of radiation background ($n=2, w=1/3$), namely that of $x=0$.

Finally, we have identified the invariant set x^2+y+

$\Sigma^2 = 1$, which contains cosmological solutions without standard matter.

C. Local analysis of the dynamical system

The system (46)-(49) admits two circles of critical points, given by $C_\epsilon : \Sigma^2 + Q^2 - 2\epsilon Q = 0$, $x + Q = \epsilon$, $y = 0$ (we use the notation $\epsilon = \pm 1$). They exist for $n \in [1, 3]$, they are located in the boundary of Ψ, and they correspond to solutions in the full phase space, satisfying $K = y = z = 0$. In order to be more transparent, let us consider the parametrization

$$C_\epsilon := \begin{cases} Q = \epsilon + \sin u \\ \Sigma = \cos u \\ x = -\sin u. \end{cases}, \; u \in [0, 2\pi] \tag{52}$$

For all the critical points we calculate the eigenvalues of the perturbation matrix $\boldsymbol{\Xi}$, which will determine their stability. These eigenvalues, evaluated at C_ϵ, are

$$\begin{aligned} &\{0, \, -2(n-1)(\cos u - 2\epsilon), \\ &\; -2(n-1)\left[(n-2)\sin u - \epsilon(3-n)\right], \\ &\; 2(n-2)\sin u + 6\epsilon(n-1)\}. \end{aligned} \tag{53}$$

For the values of u such that there exists only one zero eigenvalue, the curves are actually "normally hyperbolic"[2], and thus we can analyze the stability by analyzing the sign of the real parts of the non-null eigenvalues [48]. Therefore, we deduce that:

- No part of C_+ (C_-) is a stable (unstable). Thus, any part of C_+ (C_-) is either unstable (stable) or saddle.
- For $-\frac{1}{3} < w \leq -\frac{1}{6}$, one branch of C_- ($\frac{3(3w+1)}{3w-1} < \sin(u) \leq 1$) is stable, and one branch of C_+ ($\frac{3(3w+1)}{3w-1} < \sin(-u) \leq 1$) is unstable.
- For $-\frac{1}{6} < w < \frac{2}{3}$, the whole C_- is stable and the whole C_+ is unstable.
- For $\frac{2}{3} \leq w \leq 1$, one branch of C_- ($\frac{3(w-1)}{3w-1} < \sin(u) \leq 1$) is stable, and one branch of C_+ ($\frac{3(w-1)}{3w-1} < \sin(-u) \leq 1$) is unstable.

Note that amongst the curves C_ϵ, we can select the representative critical points labelled by $\mathcal{N}_\epsilon := (Q = 0, \Sigma = 0, x = \epsilon, y = 0)$ and $\mathcal{L}_\epsilon := (Q = 2\epsilon, \Sigma = 0, x = -\epsilon, y = 0)$ described in [36]. In our notation: $\mathcal{L}_- = C_-|_{u=3\pi/2}, \mathcal{N}_+ = C_+|_{u=3\pi/2}$ and $\mathcal{N}_- = C_-|_{u=\pi/2}, \mathcal{L}_+ = C_+|_{u=\pi/2}$. Moreover, and contrary to the investigation of [36], we obtain four new critical points, which are a pure result of the anisotropy. They are labelled by $P_1^\epsilon := (Q = \epsilon, \Sigma = \epsilon, x = 0, y = 0)$ and $P_2^\epsilon := (Q = \epsilon, \Sigma = -\epsilon, x = 0, y = 0)$, and obviously $P_1^+ = C_+|_{u=0}, P_1^- = C_-|_{u=\pi}$ and $P_2^+ = C_+|_{u=\pi}, P_2^- = C_-|_{u=0}$. Thus, it is easy to see that:

- For $-\frac{1}{6} < w < \frac{2}{3}$, $\mathcal{L}_+$ is unstable and $\mathcal{L}_-$ is stable. The critical point $\mathcal{N}_+$ ($\mathcal{N}_-$) is always unstable (stable) except in the case $w = -\frac{1}{3}$. These results match with the results obtained in [36].
- The critical points P_1^ϵ and P_2^ϵ exist always. They are non-hyperbolic, presenting an 1-dimensional center manifold tangent to the line $x + Q = 0$. $P_{1,2}^-$ ($P_{1,2}^+$) have a 3D stable (unstable) manifold provided $-\frac{1}{3} < w < 1$.

The curves of critical points C_ϵ, and the representative critical points $\mathcal{N}_\epsilon$, $\mathcal{L}_\epsilon$, P_1^ϵ and P_2^ϵ, are enumerated in Table I.

Until now we have extracted and analyzed the stability of the curves of critical points C_ϵ of the system (46)-(49), and their representative critical points. However, the system (46)-(49) admits additionally ten isolated critical points enumerated in Table II, where we also present the necessary conditions for their existence. [3]

The eigenvalues of the Jacobian matrix are presented in Appendix A and thus here we straightway provide each point's type.

- The two critical points $\mathcal{A}_\epsilon$ exist for $-\frac{1}{6} \leq w \leq 1$. The critical point $\mathcal{A}_+$ ($\mathcal{A}_-$) is a sink, that is a stable one (source, that is an unstable) provided $w_+ < w \leq 1$, where $w_+ = \frac{1}{3}\left(-2 + \sqrt{3}\right) \approx -0.09$. Otherwise they are saddle points. Equivalently, we can express the stability intervals in terms of n, using the relation (34). Thus, the aforementioned interval becomes $n_+ < n \leq 3$, where $n_+ = 3(w_+ + 1)/2 \approx 1.37$.
- The critical points $\mathcal{B}_\epsilon$ exist for $-\frac{1}{3} \leq w \leq \frac{2}{3}$. Thus, they are always saddle points having a 2-dimensional stable manifold and a 2-dimensional unstable manifold if $-\frac{1}{3} < w < \frac{2}{3}$.
- The critical points P_3^ϵ exist for $-\frac{1}{3} \leq w \leq w_+$. The are non-hyperbolic, and due to the 1-dimensional center manifold presented in the Appendix A, the stable (unstable) manifold of P_3^+ (P_3^-) is 3D for $-\frac{1}{3} < w < w_+$.

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[2] Since we are dealing with curves of critical points, every such point has necessarily at least one eigenvalue with zero real part. "Normally hyperbolic" means that the only eigenvalues with zero real parts are those whose corresponding eigenvectors are tangent to the curve [48].

[3] Strictly speaking, the system (46)-(49) admits two more critical points with coordinates $Q = \pm 2, \Sigma = \pm 1, x = 0, y = 0$. However, they are unphysical since they satisfy $|Q + x| > 1$ and thus ${}^2K \equiv \left({e_2}^2\right)^2 < 0$.

Cr./Curve	Q	Σ	x	y	Existence	Stability
$\mathcal{N}_+$	0	0	1	0	always	unstable
$\mathcal{N}_-$	0	0	-1	0	always	stable
$\mathcal{L}_+$	2	0	-1	0	always	unstable for $1.25 \leq n \leq 2.5$
$\mathcal{L}_-$	-2	0	1	0	always	stable for $1.25 \leq n \leq 2.5$
P_1^+	1	1	0	0	$1 < n \leq 3$	non-hyperbolic with $3D$ unstable manifold
P_1^-	-1	-1	0	0	$1 < n \leq 3$	non-hyperbolic with $3D$ stable manifold
P_2^+	1	-1	0	0	$1 < n \leq 3$	non-hyperbolic with $3D$ unstable manifold
P_2^-	-1	1	0	0	$1 < n \leq 3$	non-hyperbolic with $3D$ stable manifold
$\mathcal{C}_+$	$1 + \sin u$	$\cos u$	$-\sin u$	0	always	unstable for $1.25 \leq n \leq 2.5$
$\mathcal{C}_-$	$-1 + \sin u$	$\cos u$	$-\sin u$	0	always	stable for $1.25 \leq n \leq 2.5$

TABLE I: The curves of critical points C_ϵ, and the representative critical points $\mathcal{N}_\epsilon$, $\mathcal{L}_\epsilon$, P_1^ϵ and P_2^ϵ of the cosmological system (we use $\epsilon = \pm 1$). u varies in $[0, 2\pi]$. For more details on the stability of C_ϵ see the text.

Cr. P.	Q	Σ	x	y	Existence	Stability
$\mathcal{A}_+$	$\frac{2n-1}{3(n-1)}$	0	$\frac{n-2}{3(n-1)}$	$\frac{(2n-1)(4n-5)}{9(n-1)^2}$	$1.25 \leq n \leq 3$	stable for $n_+ < n < 3$ saddle for $1.25 \leq n \leq n_+$
$\mathcal{A}_-$	$-\frac{2n-1}{3(n-1)}$	0	$-\frac{n-2}{3(n-1)}$	$\frac{(2n-1)(4n-5)}{9(n-1)^2}$	$1.25 \leq n \leq 3$	unstable for $n_+ < n < 3$ saddle for $1.25 \leq n \leq n_+$
$\mathcal{B}_+$	$\frac{1}{3-n}$	0	$\frac{n-2}{n-3}$	0	$1 < n \leq 2.5$	saddle
$\mathcal{B}_-$	$-\frac{1}{3-n}$	0	$-\frac{n-2}{n-3}$	0	$1 < n \leq 2.5$	saddle
P_3^+	$\frac{1}{2-n}$	0	$-\frac{n-1}{2-n}$	$\frac{n-1}{n(n-2)^2}$	$1 < n \leq n_+$	non-hyperbolic with $3D$ stable manifold
P_3^-	$-\frac{1}{2-n}$	0	$\frac{n-1}{2-n}$	$\frac{n-1}{n(n-2)^2}$	$1 < n \leq n_+$	non-hyperbolic with $3D$ unstable manifold
P_4^+	$\frac{2n^2-5n+5}{7n^2-16n+10}$	$-\frac{2n^2-2n-1}{7n^2-16n+10}$	$\frac{3(n-1)(n-2)}{7n^2-16n+10}$	$\frac{9(4n^2-10n+7)(n-1)^2}{(7n^2-16n+10)^2}$	$n_+ \leq n \leq 3$	saddle with $3D$ stable manifold
P_4^-	$-\frac{2n^2-5n+5}{7n^2-16n+10}$	$\frac{2n^2-2n-1}{7n^2-16n+10}$	$-\frac{3(n-1)(n-2)}{7n^2-16n+10}$	$\frac{9(4n^2-10n+7)(n-1)^2}{(7n^2-16n+10)^2}$	$n_+ \leq n \leq 3$	saddle with $3D$ unstable manifold
P_5^+	$\frac{1}{n-2}$	$\frac{\sqrt{2n-5}}{n-2}$	$\frac{n-3}{n-2}$	0	$2.5 \leq n \leq 3$	non-hyperbolic, with a 2D center manifold
P_5^-	$-\frac{1}{n-2}$	$\frac{\sqrt{2n-5}}{n-2}$	$-\frac{n-3}{n-2}$	0	$2.5 \leq n \leq 3$	non-hyperbolic, with a 2D center manifold

TABLE II: The isolated critical points of the cosmological system. We use the notation $n_+ = \frac{1+\sqrt{3}}{2} \approx 1.37$.

- The critical points P_4^ϵ exist for $w_+ \leq w \leq 1$. P_4^+ (P_4^-) has a 3D stable (unstable) manifold and a 1D unstable (stable) manifold provided $w_+ < w \leq 1$. Thus, they are always saddle points.

- The critical points P_5^ϵ exist for $\frac{2}{3} \leq w \leq 1$. They are non-hyperbolic, there exists a 2D center manifold and P_5^+ (P_5^-) has a 2D unstable (stable) manifold.

Lastly, as we have mentioned, in this subsection we have investigated the system (46)-(49), leaving outside the decoupled equation (50). However, for completeness, we study equation (50), and especially the stability across the direction ${E_1}^1$, in Appendix B.

D. Physical description of the solutions and connection with observables

Let us now present the formalism of obtaining the physical description of a critical point, and also connecting with the basic observables relevant for a physical discussion. These will allow us to describe the cosmological behavior of each critical point, in the next section.

Firstly, around a critical point we obtain first-order expansions for ${e_1}^1, {}^2K$, ρ_m and R in terms of τ, considering the versions of equations (22), (21) and (39), respectively given by

$$\begin{aligned} \left({e_1}^1\right)' &= -(n-1)\left[Q^* - 2\Sigma^*\right]{e_1}^1, \\ \left({}^2K\right)' &= -2(n-1)\left[Q^* + \Sigma^*\right]{}^2K, \\ \rho_m' &= -2n(n-1)Q^*\rho_m, \\ R' &= 2x^*R, \end{aligned} \tag{54}$$

where the star-upperscript denotes the evaluation at a specific critical point, and the prime denotes derivative with respect to τ. The last equation follows from the definition of x given by (40). In general we can consider the case $x^* \neq 0$ and $y^\star \neq 0$, since in the simple case of $x^* = 0$ and $y^\star \neq 0$ we obtain $R = R_0$, with R_0 a constant, while in the case $y^\star = 0$ we acquire $R = 0$.

In order to express the above determined functions of τ in terms of the comoving time variable t, we invert the solution of

$$\frac{dt}{d\tau} = \frac{n-1}{D^*} \tag{55}$$

with D^* being the first-order solution of

$$D' = D(n-1)\Upsilon^\star, \tag{56}$$

where

$$\Upsilon^\star = x^* - \Sigma^* + (Q^* + x^*)\left[-3{x^*}^2 + \Sigma^* x^* \right. \\ \left. +(Q^* - 3\Sigma^*)\Sigma^* + n\left({x^*}^2 + {\Sigma^*}^2 + y^* - 1\right)\right]. \tag{57}$$

Solving equations (55)-(56) (with initial conditions $D(0) = D_0$ and $t(0) = t_0$) we obtain

$$t(\tau) = \frac{1 - e^{(1-n)\tau\Upsilon^*}}{D_0\Upsilon^*} + t_0. \tag{58}$$

Thus, inverting the last equation for τ and substituting in the solution of (54) with initial conditions ${e_1}^1(0) = {e_1}^1_0, {}^2K(0) = {}^2K_0, \rho_m(0) = \rho_{m0}, R(0) = R_0$, we acquire

$$\begin{aligned} {e_1}^1(t) &= {e_1}^1_0(D_0(t_0 - t)\Upsilon^* + 1)^{\frac{Q^* - 2\Sigma^*}{\Upsilon^*}}, \\ {}^2K(t) &= {}^2K_0(D_0(t_0 - t)\Upsilon^* + 1)^{\frac{2(Q^* + \Sigma^*)}{\Upsilon^*}} \\ \rho_m(t) &= \rho_{m0}(D_0(t_0 - t)\Upsilon^* + 1)^{\frac{2nQ^*}{\Upsilon^*}}, \\ R(t) &= R_0(D_0(t_0 - t)\Upsilon^* + 1)^{\frac{2x^*}{(1-n)\Upsilon^*}}. \end{aligned} \tag{59}$$

Finally, it can be shown that the length scale ℓ along the flow lines, defined in (5), can be expressed as [34]

$$\ell = \ell_0(\ell_1 - t\Upsilon^*)^{-\frac{Q^*}{\Upsilon^*}}, \tag{60}$$

where $\ell_0 = \left[\left({e_1}^1_0\right)\left({}^2K_0\right)\right]^{-\frac{1}{3}}$ and $\ell_1 = D_0 t_0 \Upsilon^* + 1$. In summary, expressions (59) and (60) determine the solution, that is the evolution of various quantities, at a critical point.

Let us now come to the observables. Using the above expressions, we can calculate the deceleration parameter q defined as usual as [34]

$$q = -\frac{\ell\ddot{\ell}}{(\dot{\ell})^2}. \tag{61}$$

Additionally, we can calculate the effective (total) equation-of-state parameter of the universe $w_{\rm eff}$, defined conventionally as

$$w_{\rm eff} \equiv \frac{P_{\rm tot}}{\rho_{\rm tot}} = \frac{\frac{p_m}{f'(R)} + P}{\frac{\rho_m}{f'(R)} + \mu}, \tag{62}$$

where $P_{\rm tot}$ and $\mu_{\rm tot}$ are respectively the total isotropic pressure and the total energy density as they can be read from equation (18), where P and μ are given by (16). Therefore, in terms of the auxiliary variables we have

$$q = \frac{\Sigma^2 - x(2Q + x) + 1}{Q^2} - \frac{ny}{(n-1)Q^2} \tag{63}$$

$$w_{\rm eff} = \frac{2ny}{3(n-1)\left(x^2 + 2Qx + \Sigma^2 - 1\right)} + \frac{1}{3}. \tag{64}$$

Finally, the various density parameters defined in (23)-(26), in terms of the auxiliary variables straightforwardly read

$$\begin{aligned} \Omega_k &= \frac{(Q+x)^2 - 1}{Q^2} \\ \Omega_m &= \frac{1 - x^2 - y - \Sigma^2}{Q^2} \\ \Omega_{\rm curv.fl} &= \frac{y - 2xQ}{Q^2} \\ \Omega_\sigma &= \left(\frac{\Sigma}{Q}\right)^2. \end{aligned} \tag{65}$$

In conclusion, at each critical point we can calculate the values of the basic observables q, $w_{\rm eff}$ and the various density parameters, and also calculate the specific physical solution, that is obtain the behavior of $\ell(t)$, $\rho_m(t)$ and $R(t)$.

Finally, it is interesting to notice that in Kantowski-Sachs geometry one can easily handle isotropization. In particular, the geometry becomes isotropic if σ_+ becomes zero, as can be seen from (1) and (9). Thus, critical points with $\Sigma = 0$ (or more physically $\Omega_\sigma = 0$) correspond to Friedmann points, that is to isotropic universes, and when such an isotropic point is an attractor then we obtain asymptotic isotropisation in the future [31].

IV. COSMOLOGICAL IMPLICATIONS

In the previous sections we formulated $f(R)$-gravity in the case of the homogeneous and anisotropic Kantowski-Sachs geometry, focusing on the $f(R) = R^n$ ansatz, and we performed a detailed phase-space analysis. Thus, in the present section we discuss the physical implications of the mathematical results, focusing on the physical behavior and on observable quantities.

First of all, for each critical point of Tables I and II we calculate the effective (total) equation-of-state parameter of the universe $w_{\rm eff}$ using (64), and the deceleration parameter q using (63), and the values of the density parameters Ω_k, Ω_m, $\Omega_{\rm curv.fl}$, and Ω_σ using (65). These results are presented in the Table III.

Furthermore, for each critical point we use (59) and (60), in order to extract the behavior of the physically important quantities $\ell(t)$, $\rho_m(t)$ and $R(t)$ at this critical point. As we have mentioned $\ell(t)$ is the length scale along the flow lines and in the case of zero anisotropy (for instance in FRW cosmology) it is just the usual scale factor. Additionally, $\rho_m(t)$ is the matter energy density and $R(t)$ is the Ricci scalar. These solutions are presented in the last column of Tables IV and V.

Finally, in the last column of Tables IV and V we also present the physical description of the corresponding solution, taking into account all the above information. In particular, since the auxiliary variable Q defined in (40) is the Hubble scalar divided by a positive constant, $Q > 0$

Cr.P.	$w_{\rm eff}$	q	Ω_k	Ω_m	$\Omega_{\rm curv.fl}$	Ω_σ
$\mathcal{A}_\pm$	$-\frac{6n^2-7n-1}{3(n-1)(2n-1)}$	$-\frac{2n^2-2n-1}{(n-1)(2n-1)}$	0	0	1	0
$\mathcal{B}_\pm$	$\frac{1}{3}$	1	0	$5-2n$	$2(n-2)$	0
$P_1^\pm, P_2^\pm$	$\frac{1}{3}$	2	0	0	0	1
$P_3^\pm$	$-\frac{1}{3}$	0	0	$-2n+2+\frac{1}{n}$	$2n-1-\frac{1}{n}$	0
$P_4^\pm$	$\frac{n(4(9-4n)n-21)-1}{3n(8(n-3)n+21)-3}$	$\frac{6-3n}{n(2n-5)+5}-1$	$-\frac{3\left(2n^2-2n-1\right)\left(4n^2-10n+7\right)}{\left(2n^2-5n+5\right)^2}$	0	$\frac{3(n-1)\left(8n^3-24n^2+21n-1\right)}{\left(2n^2-5n+5\right)^2}$	$\frac{\left(2n^2-2n-1\right)^2}{\left(2n^2-5n+5\right)^2}$
$P_5^\pm$	$\frac{1}{3}$	$2(n-2)$	0	0	$2(3-n)$	$2n-5$
$C_\pm$	$\frac{1}{3}$	$\frac{2}{1\pm\sin(u)}$	0	0	$\frac{2\sin(u)}{\sin(u)\pm1}$	$\frac{\cos^2(u)}{(\sin(u)\pm1)^2}$

TABLE III: The values of the basic observables, namely the effective (total) equation-of-state parameter $w_{\rm eff}$, the deceleration parameter q, the curvature density parameter Ω_k, the matter density parameter Ω_m, the "curvature-fluid" density parameter $\Omega_{\rm curv.fl}$ and the shear density parameter Ω_σ, defined in (63)-(65), at the isolated critical points and curves of critical points of the cosmological system. We display the information about the non-isolated critical points $P_1^\pm, P_2^\pm$ to emphasize they are solutions dominated by shear.

corresponds to an expanding universe, while $Q<0$ to a contracting one. Furthermore, as usual, for an expanding universe $q<0$ corresponds to accelerating expansion and $q>0$ to decelerating expansion, while for a contracting universe $q<0$ corresponds to decelerating contraction and $q>0$ to accelerating contraction. Additionally, if $w_{\rm eff}<-1$ then the total equation-of-state parameter of the universe exhibits phantom behavior. Lastly, critical points with $\Sigma=0$ correspond to isotropic universes.

Let us analyze the physical behavior in more details. The critical point $\mathcal{A}_+$ is stable for $1.37 \lesssim n < 3$, and thus at late times the universe can result in it. It corresponds to an isotropic expanding universe, in which the expansion is accelerating. Additionally, since the curvature energy density is zero, this point corresponds to an asymptotically flat universe (this is equivalent to close or open FRW universes which may possess zero curvature solutions as asymptotic states). Furthermore, in the case $2<n<3$ the total equation-of-state parameter of the universe exhibits phantom behavior. Thus, the anisotropic Kantowsky-Sachs R^n-gravity can lead the universe not only to be accelerating, but also to be in the phantom "phase", a result of great cosmological interest. We have to mention here that since point $\mathcal{A}_+$ is asymptotically an FRW one, one expects to obtain phantom behavior in isotropic (FRW) R^n-gravity too. Although in the phase-space analysis of FRW R^n-gravity [36], the authors have focused only on acceleration, without examining the total equation-of-state parameter of the universe, it is easy to see that if ones defines it and examines its features he will find phantom behavior in that case too. This is also the case in Bianchi I and Bianchi III R^n-gravity [31, 49], where the authors would have found the phantom behavior if they had calculated the total equation-of-state parameter. Therefore, we conclude that the phantom behavior is a result of the modification of gravity, as it has been discussed in detail in the literature [21–23]. Finally, in the case where the matter is dust ($w=0$, that is $n=3/2$), $\rho_m(t)$ behaves like $\ell(t)^{-3}$ as expected, and this acts as a self-consistency test for our analysis. Additionally, note that in the special case $n=2$, that is when $w=1/3$, i.e when radiation dominates the universe, the aforementioned stable solution corresponds to a de-Sitter expansion (this is not a new feature since de-Sitter solutions are known to exist in Bianchi I and Bianchi III R^n-gravity [31]). This is of great significance since such a behavior can describe the inflationary epoch of the universe.

In the above critical point the isotropization has been achieved. Such late-time isotropic solutions, than can attract an initially anisotropic universe, are of significant cosmological interest and have been obtained and discussed in the literature [14, 31]. The acquisition of such a solution was one of the motives of the present work, as of many works on anisotropic cosmologies, since, as we discussed in the Introduction, it is the only robust approach in confronting isotropy of standard cosmology. The fact that this solution is accompanied by acceleration or phantom behavior, makes it a very good candidate for the description of the observable universe.

The critical point $\mathcal{A}_-$ which corresponds to an isotropic contracting universe is not stable and thus it cannot be the late-time state of the universe. Similarly, the points $\mathcal{B}_+$ and $\mathcal{B}_-$, which correspond to isotropic expanding and contracting universes respectively, and in which the total matter/energy mimics radiation, are saddle points and thus they cannot be the late-time solution for the universe, too. The points P_5^ϵ, which correspond to decelerating expansion (for $\epsilon=+1$) and to accelerating contraction (for $\epsilon=-1$), are non-hyperbolic with a 2D center manifold, and thus, generically, the universe cannot be led to them.

The critical points P_1^- and P_2^- which correspond to accelerating contraction, in which the total matter/energy behaves like radiation, are non-hyperbolic critical points, but they do possess a $3D$ stable manifold. By an explicit and straightforward computation of their center manifolds [50] we deduce that for $2<n<3$ each center manifold is locally asymptotically stable (the equation governing the dynamics on the center manifold is a gradient-type differential equation with potential function having a degenerate local minimum at the origin), and thus P_1^-

Cr.P.	Υ^*	Solution/description
$\mathcal{A}_+$	$\frac{(n-2)}{3(n-1)^2}$	$\ell(t)=\begin{cases}\ell_0(\ell_1-t\Upsilon^*)^{s_1} & n\neq 2\\ \ell_0 e^{H_1(t-t_0)} & n=2\end{cases}$, $\rho_m(t)=\rho_{m0}\left[\frac{\ell}{\ell_0}\right]^{-2n}$, $R(t)=\begin{cases}\frac{R_0}{(\ell_1-t\Upsilon^*)^2} & n\neq\{1.25,2\}\\ R_0 & n=2\\ 0 & n=1.25\end{cases}$ Isotropic. Expanding. Accelerating for $n_+<n<3$. Phantom for $2<n<3$. For $n=2$ exhibits dS behavior
$\mathcal{A}_-$	$-\frac{(n-2)}{3(n-1)^2}$	$\ell(t)=\begin{cases}\ell_0(\ell_1-t\Upsilon^*)^{s_1} & n\neq 2\\ \ell_0 e^{H_1(t-t_0)} & n=2\end{cases}$, $\rho_m(t)=\rho_{m0}\left[\frac{\ell}{\ell_0}\right]^{-2n}$, $R(t)=\begin{cases}\frac{R_0}{(\ell_1-t\Upsilon^*)^2} & n\neq\{1.25,2\}\\ R_0 & n=2\\ 0 & n=1.25\end{cases}$ Isotropic. Contracting. Decelerating for $n_+<n<3$. Phantom for $2<n<3$. For $n=2$ exhibits collapsing AdS behavior
$\mathcal{B}_+$	$\frac{2}{n-3}$	$\ell(t)=\ell_0\sqrt{(\ell_1-t\Upsilon^*)}$, $\rho_m(t)=\rho_{m0}(\ell_1-t\Upsilon^*)^{-n}$, $R(t)=0$ Isotropic. Expanding. Decelerating. Total matter/energy mimics radiation.
$\mathcal{B}_-$	$-\frac{2}{n-3}$	$\ell(t)=\ell_0\sqrt{(\ell_1-t\Upsilon^*)}$, $\rho_m(t)=\rho_{m0}(\ell_1-t\Upsilon^*)^{-n}$, $R(t)=0$ Isotropic. Contracting. Accelerating. Total matter/energy mimics radiation.
P_3^+	$-\frac{1}{n-2}$	$\ell(t)=\ell_0(\ell_1-t\Upsilon^*)=a_1+a_2t$, $\rho_m(t)=\rho_{m0}(\ell_1-t\Upsilon^*)^{-2n}$, $R(t)=\frac{R_0}{(\ell_1-t\Upsilon^*)^2}$ Isotropic. Expanding. Zero acceleration.
P_3^-	$\frac{1}{n-2}$	$\ell(t)=\ell_0(\ell_1-t\Upsilon^*)=a_1+a_2t$, $\rho_m(t)=\rho_{m0}(\ell_1-t\Upsilon^*)^{-2n}$,$R(t)=\frac{R_0}{(\ell_1-t\Upsilon^*)^2}$ Isotropic. Contracting. Zero acceleration.
P_4^+	$\frac{3(n-2)}{7n^2-16n+10}$	$\ell(t)=\begin{cases}\ell_0(\ell_1-t\Upsilon^*)^{s_2} & n\neq 2\\ \ell_0 e^{H_2(t-t_0)} & n=2\end{cases}$, $\rho_m(t)=\rho_{m0}\left[\frac{\ell}{\ell_0}\right]^{-2n}$, $R(t)=\begin{cases}\frac{R_0}{(\ell_1-t\Upsilon^*)^2} & n\neq 2\\ R_0, & n=2\end{cases}$ Expanding. Accelerating for $n_+<n<3$. Phantom for $2<n<M_+$. For $n=2$ exhibits dS behavior
P_4^-	$-\frac{3(n-2)}{7n^2-16n+10}$	$\ell(t)=\begin{cases}\ell_0(\ell_1-t\Upsilon^*)^{s_2} & n\neq 2\\ \ell_0 e^{H_2(t-t_0)} & n=2\end{cases}$, $\rho_m(t)=\rho_{m0}\left[\frac{\ell}{\ell_0}\right]^{-2n}$, $R(t)=\begin{cases}\frac{R_0}{(\ell_1-t\Upsilon^*)^2} & n\neq 2\\ R_0, & n=2\end{cases}$ Contracting. Decelerating for $n_+<n<3$. Phantom for $2<n<M_+$. For $n=2$ exhibits collapsing AdS behavior
P_5^+	$\frac{(3-2n)}{n-2}$	$\ell(t)=\ell_0(\ell_1-t\Upsilon^*)^{s_3}$, $\rho_m(t)=\rho_{m0}\left[\frac{\ell}{\ell_0}\right]^{-2n}$, $R(t)=0$ Expanding. Decelerating. Total matter/energy mimics radiation.
P_5^-	$-\frac{(3-2n)}{n-2}$	$\ell(t)=\ell_0(\ell_1-t\Upsilon^*)^{s_3}$, $\rho_m(t)=\rho_{m0}\left[\frac{\ell}{\ell_0}\right]^{-2n}$, $R(t)=0$ Contracting. Accelerating. Total matter/energy mimics radiation.

TABLE IV: The behavior of $\ell(t)$ (length scale along the flow lines), of $\rho_m(t)$ (matter energy density) and of $R(t)$ (Ricci scalar) at the critical points of the cosmological system. We use the notations $s_1=-\frac{(n-1)(2n-1)}{n-2}$, $s_2=-\frac{2n^2-5n+5}{3(n-2)}$, $s_3=\frac{1}{2n-3}$, $p_\epsilon(u)=\frac{\epsilon+\sin(u)}{3\epsilon+\sin(u)}$, $n_+=\frac{1+\sqrt{3}}{2}\approx 1.37$ and $M_+=\frac{1}{4}\left(5+\sqrt{21}\right)\approx 2.40$.

Cr.P.	Υ^*	Solution/description
C_+	$-\left[3+\sin(u)\right]$	$\ell(t)=\ell_0(\ell_1-t\Upsilon^*)^{p_+(u)}$, $\rho_m(t)=\rho_{m0}\left[\frac{\ell}{\ell_0}\right]^{-2n}$, $R(t)=0$ Expanding. Decelerating. Total matter/energy mimics radiation.
C_-	$-\left[-3+\sin(u)\right]$	$\ell(t)=\ell_0(\ell_1-t\Upsilon^*)^{p_-(u)}$, $\rho_m(t)=\rho_{m0}\left[\frac{\ell}{\ell_0}\right]^{-2n}$, $R(t)=0$ Contracting. Accelerating. Total matter/energy mimics radiation.

TABLE V: Physical behavior of the solutions at the curves of critical points of the cosmological system. We use the notation $p_\epsilon(u)=\frac{\epsilon+\sin(u)}{3\epsilon+\sin(u)}$.

and P_2^- are locally asymptotically stable [50], and can attract the universe at late times. On the contrary, for $1 < n < 2$, P_1^- and P_2^- are locally asymptotically unstable (of saddle type). In summary, although P_1^- and P_2^- are not stable, they do have a significant probability to be a late-time state for the universe (this is realized for initial conditions on its stable manifold), or at least the universe can stay near these solutions for a long time before the dynamics remove it from them. On the other hand, the points P_1^+ and P_2^+ which correspond to decelerating expansion, possess a $3D$ unstable manifold, and thus they cannot be a late-time solution of the universe.

Non-hyperbolic critical point with a $3D$ stable manifold is also the critical point P_3^+, which corresponds to an asymptotically flat isotropic expansion with zero acceleration, and thus it has also a large probability to be a late-time state of the universe. However, note that this solution corresponds to zero acceleration, and thus $\ell(t)$ is a linear function of t. On the other hand, the critical point P_3^- (isotropic, contracting with zero acceleration) possesses a $3D$ unstable manifold and thus it cannot attract the universe at late times.

The critical point P_4^+ is saddle with a $3D$ stable manifold, and thus it has a large probability to be a late-time state of the universe. It corresponds to a non-flat, accelerating expansion, for $1.37 \lesssim n < 3$, and furthermore in the case $2 < n < 3$ it exhibits phantom behavior. Finally, for $n = 2$, that is for $w = 1/3$, it corresponds to a de-Sitter expansion. On the other hand P_4^- (decelerating contraction) is highly unstable and therefore it cannot attract the universe at late times.

We mention here that the critical points P_1^ϵ,P_2^ϵ,P_3^ϵ,P_4^ϵ and P_5^ϵ are not present in isotropic (FRW) R^n-gravity, as compared with [36]. They arise as a pure result of the anisotropy, and this shows that the much more complicated structure of anisotropic geometries leads to radically different cosmological behaviors comparing to the simple isotropic scenarios.

Additionally, we have to analyze the behavior of the curves of critical points C_ϵ. The points C_-, which correspond to accelerating contraction, are stable if $1.25 \leq n \leq 2.5$, and thus they can be late-time solutions of the universe. On the other hand, C_+, which correspond to decelerating expansion, are unstable and thus they cannot attract the universe at late times.

Let us finish the physical discussion by referring to static solutions, in order to compare to the FRW case of [36]. From the cosmological point of view, static solutions possess $\ell(t) = const.$, that is $\dot{\ell}(t) = 0$ and $\ddot{\ell}(t)$=0, in order to obtain $H(t) = 0$ and $\dot{H}(t) = 0$ (note that one needs both conditions, since in a cosmological bounce or turnaround, that is when a universe changes from contracting to expanding or vice versa, $\dot{\ell}(t)$ is zero instantly, but then it becomes positive or negative again). Thus, using our auxiliary variable Q defined in (40) static solutions should have $Q = 0$ and $\dot{Q} = 0$. In conclusion, since in all the aforementioned critical points $\dot{Q} = 0$ by definition, static solutions are just those with $Q = 0$. At this point there is another important difference comparing to the isotropic case of [36]. In particular, in FRW geometry, the Ricci scalar is $R = 6(\dot{H} + 2H^2)$, and therefore static solutions have $R = 0$ (however the inverse is not true, that is not all solutions with $R = 0$ correspond to static ones, since one can have $R = 0$ but with $\dot{H}$ and H non-zero). On the other hand, in the anisotropic case R is given by (28), i.e $R = 12H^2 + 6\sigma_+^2 + 6\dot{H} + 2\,{}^2K$, that is we have also the presence of additional terms, and therefore static solutions do not correspond to $R = 0$ unless σ_+ and ${}^2K$ are also zero, which is not fulfilled in general. Thus, in the anisotropic case one cannot use R, or equivalently the auxiliary variable y defined in (40), in order to straightway determine the static solutions, in contrast to the isotropic case [36] where the authors use $y = 0$ for such a determination.

Now, in order to present the aforementioned results in a more transparent way, we perform a numerical elaboration of our cosmological system, using a seventh-eighth order continuous Runge-Kutta method with absolute error 10^{-8}, and relative error 10^{-8} [51]. In Fig. 1 we depict some orbits in the invariant set $y = 0$, in the case of dust matter ($w = 0, n = 3/2$). As we observe,

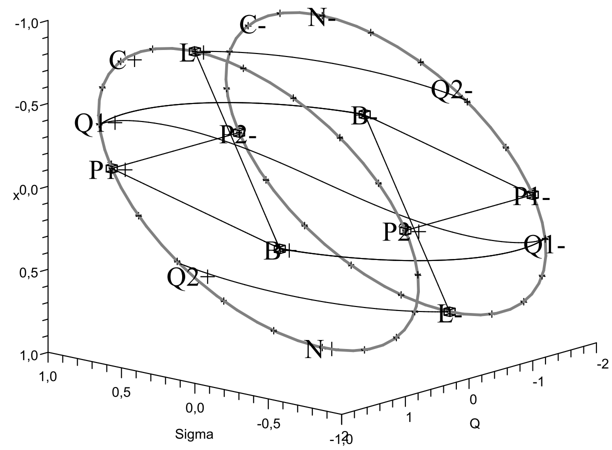

FIG. 1: Projection of the phase space on the invariant set $y = 0$, in the case of dust matter ($w = 0, n = 3/2$). The critical points $Q_{1,2}^\epsilon$ have coordinates $Q_1^\epsilon := (Q = 4\epsilon/3, \Sigma = 2\sqrt{2}\epsilon/3, -\epsilon/3)$ and $Q_2^\epsilon := (Q = \epsilon/3, \Sigma = \sqrt{5}\epsilon/3, 2\epsilon/3)$. All the points in the circle C_+ are unstable whereas all the points in the circle C_- are stable. The heteroclinic sequences reveal the possible transition from expansion to contraction and vice versa (see text).

we have the appearance of four critical points with coordinates $Q_1^\epsilon := (Q = 4\epsilon/3, \Sigma = 2\sqrt{2}\epsilon/3, -\epsilon/3)$ and $Q_2^\epsilon := (Q = \epsilon/3, \Sigma = \sqrt{5}\epsilon/3, 2\epsilon/3)$, located in the invariant curves C^ϵ. All the points in the circle C_+ are

unstable whereas all the points in the circle C_- are stable. Note that the critical points P_3^ϵ coincide with $\mathcal{N}_\epsilon$. In the figure we also display heteroclinic sequences [38] of types

$$\begin{aligned}
&Q_1^+ \longrightarrow \begin{cases} \mathcal{B}^- \longrightarrow \mathcal{L}_- \\ \mathcal{B}^- \longrightarrow P_1^- \\ Q_1^- \end{cases} \\
&\mathcal{L}_+ \longrightarrow \begin{cases} \mathcal{B}^+ \longrightarrow Q_1^- \\ Q_2^- \end{cases} \\
&P_1^+ \longrightarrow \begin{cases} \mathcal{B}^+ \longrightarrow Q_1^- \\ P_2^- \end{cases} \\
&P_2^+ \longrightarrow P_1^- \\
&Q_2^+ \longrightarrow \mathcal{L}_-.
\end{aligned} \tag{66}$$

Thus, we can have evolutions in which the + branch and − branch are connected, that is we can have the transition from expansion to contraction and vice versa. This is just a cosmological turnaround and a cosmological bounce, and their realization in the present scenario reveals the capabilities of the model. It is interesting to note that such behaviors can be realized in FRW R^n-gravity [40, 52], however they are impossible in General Relativity Kantowsky-Sachs cosmology [45]. Therefore, we conclude that they are a result of the R^n gravitational sector and not of the anisotropy.

In Fig. 2 we display some orbits in the case of radiation ($w = 1/3, n = 2$), where as we mentioned in subsection III B the invariant set $x = 0$ appears. Particularly, we observe the existence of heteroclinic sequences of types

$$\begin{aligned}
&\mathcal{A}_- \longrightarrow \begin{cases} P_4^- \longrightarrow P_4^+ \longrightarrow \mathcal{A}_+ \\ \mathcal{B}^- \end{cases} \\
&\mathcal{B}_+ \longrightarrow \mathcal{A}_+ \\
&P_1^- \longrightarrow P_4^+ \longrightarrow P_2^+ \\
&P_2^- \longrightarrow P_4^- \longrightarrow P_1^+ \\
&P_1^+ \longrightarrow P_2^- \\
&P_2^+ \longrightarrow P_1^-,
\end{aligned} \tag{67}$$

revealing the realization of a cosmological bounce or a cosmological turnaround. Similarly to the isotropic case (see figure 5 of [36]) there is one orbit of type $\mathcal{B}_+ \longrightarrow \mathcal{A}_+$ and one of type $\mathcal{A}_- \longrightarrow \mathcal{B}_-$. However, in the present case we have the additional existence of an heteroclinic sequence of type $\mathcal{A}_- \longrightarrow P_4^- \longrightarrow P_4^+ \longrightarrow \mathcal{A}_+$, corresponding to the transition from collapsing AdS to expanding dS phase, that is we obtain a cosmological bounce followed by a de Sitter expansion, which could describe the inflationary stage. This significant behavior is a pure result of the anisotropy and reveals the capabilities of the scenario. Lastly, note the very interesting possibility, of the eternal transition $P_1^- \longrightarrow P_2^+ \longrightarrow P_1^- \longrightarrow P_2^+ \cdots$ which is just the realization of cyclic cosmology [53]. Bouncing solutions are found to exist both in FRW R^n-gravity [54], as well as in the Bianchi I and Bianchi III R^n-gravity [31] (see also [55]), and thus they arise from the R^n gravitational sector. However, in the present Kantowsky-Sachs geometry cyclicity seems to be realized relatively easily (however without accompanied by isotropization), that is without fine-tuning the model-parameters, which is an advantage of the scenario.

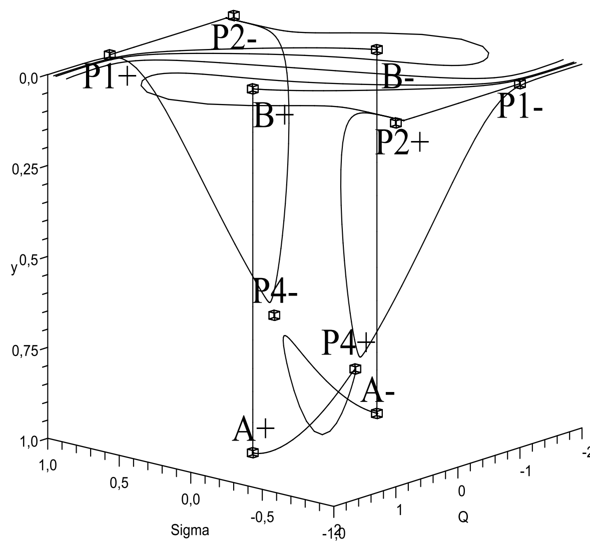

FIG. 2: Projection of the phase space on the invariant set $x = 0$, in the case of radiation ($w = 1/3, n = 2$). There is one orbit of type $\mathcal{B}_+ \longrightarrow \mathcal{A}_+$ and one of type $\mathcal{A}_- \longrightarrow \mathcal{B}_-$. The existence of an heteroclinic sequence of type $\mathcal{A}_- \longrightarrow P_4^- \longrightarrow P_4^+ \longrightarrow \mathcal{A}_+$ corresponds to a cosmological bounce.

Finally, in Fig. 3 we have extracted orbits located at $x = y = 0$, in the case of radiation. The shaded region corresponds to the unphysical portion of the phase plane (note however that this region is not invariant, since an open set of orbits enter/abandon the unphysical boundary, and thus such evolutions have to be excluded too). As we observe, in this figure the last two heteroclinic sequences of (67) are displayed.

Let us make a comment here by referring to the cosmological epoch sequence. As we mentioned in subsection II D, in the scenario at hand we can obtain the transition from the matter-dominated era to the accelerated one. However, note that the dynamical system analysis can only give analytical results relating to the late-time states of the universe. The precise evolution of the universe towards such late-time attractors depends on the initial conditions, and it can only be investigated through a detailed numerical elaboration similar to the partial one that we performed in order to produce the aforementioned three figures. Thus, by suitably determining the initial conditions, one can obtain universes that start with a de-Sitter expansion, then transit to the matter-dominated era, and finally falling into the late

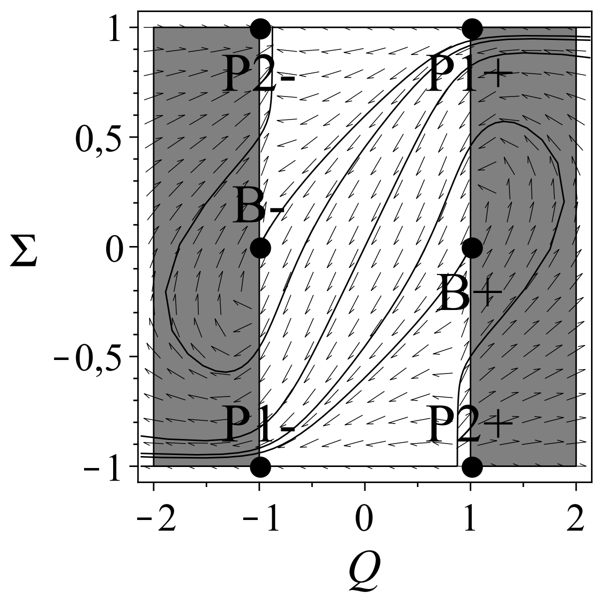

FIG. 3: Invariant set $x = 0$, $y = 0$ in the case of radiation ($w = 1/3, n = 2$). The shaded region corresponds to the unphysical portion of the phase plane. P_2^- (P_2^+) is the local future (past) attractor, while B_+, B_- and P_1^+, P_1^- are saddle points.

time accelerating solution. The detailed examination of such behaviors, along with the investigation of the basins of attraction of the various evolutions in terms of the initial conditions, and the estimation of the measures of the corresponding sub-spaces of the phase space, lie outside of the present work and are left for a future project.

We close this section by discussing the cosmological evolution in the special case where ${E_1}^1 = 0$ or $K = 0$, since in this case the Kantowsky-Sachs metric (in a comoving frame)

$$ds^2 = -\frac{(n-1)^2}{D^2}d\tau^2 + \frac{1}{D^2\left({E_1}^1\right)^2}dr^2 + \frac{1}{3D^2K}[d\vartheta^2 + \sin^2\varphi\, d\varphi^2], \quad (68)$$

is singular. Although these points are unphysical their neighbouring solutions could have physical meaning, which can be extracted by obtaining first-order evolution rates valid in an small neighborhood of the critical points of the system. In particular, we first evaluate the perturbation-matrix $\mathbf{\Xi}$ at the critical point of interest, we diagonalize it and we obtain orders of magnitude for linear combinations of the vector components $\left({E_1}^1, Q, \Sigma, x, y\right)^T$. Thus, the desired expansion can be obtained by taking the inverse linear transformation, and finally we preserve only the leading order terms as $\tau \to -\infty$ or as $\tau \to \infty$. Although this procedure is straightforward in the case of dust matter ($w = 0$, $n = 3/2$) and radiation ($w = 1/3$, $n = 2$) we do not present it explicitly since we desire to remain in the general case of ${E_1}^1 \neq 0$ and $K \neq 0$.

V. CONCLUSIONS

In this work we constructed general anisotropic cosmological scenarios where the gravitational sector belongs to the extended $f(R)$ type, and we focused on Kantowski-Sachs geometries in the case of R^n-gravity. We performed a detailed phase-space analysis, extracting the late-time solutions, and we connected the mathematical results with physical behaviors and observables.

As we saw, the universe at late times can result to a state of accelerating expansion, and additionally, for a particular n-range ($2 < n < 3$) it exhibits phantom behavior. Additionally, the universe has been isotropized, independently of the anisotropy degree of the initial conditions, and it asymptotically becomes flat. The fact that such features are in agreement with observations [1, 16] is a significant advantage of the model. Moreover, in the case of radiation ($n = 2$, $w = 1/3$) the aforementioned stable solution corresponds to a de-Sitter expansion, and it can also describe the inflationary epoch of the universe.

Note that at first sight the above behavior could be ascribed to the cosmic no-hair theorem [56], which states that a solution of the cosmological equations, with a positive cosmological constant and under the perfect-fluid assumption for matter, converges to the de Sitter solution at late times. However, we mention that such a theorem holds for matter-fluids less stiff than radiation, but more importantly it has been elaborated for General Relativity [57], without a robust extension to higher order gravitational theories [58]. In our work we extracted our results without relying at all on the cosmic no-hair theorem, which is a significant advantage of the analysis.

Apart from the above behavior, in the scenario at hand the universe has a large probability to remain in a phase of (isotropic or anisotropic) decelerating expansion for a long time, before it will be attracted by the above global attractor at late times, and this acts as an additional advantage of the model, since it is in agreement with the observed cosmological behavior. However, the precise duration of such a transient phase, unlike the attractor behavior, does depend on the initial conditions of the universe, which have to be suitable determined in order to lead to a deceleration-duration of the order of 10^9-10^{10} years, before the universe pass to the accelerating phase. Such an analysis can only arise through an explicit numerical elaboration, that is beyond the analytical, dynamical-system treatment which is where the present work focuses.

The Kantowski-Sachs anisotropic R^n-gravity can also lead to contracting solutions, either accelerating or decelerating, which are not globally stable. Thus, the universe can remain near these states for a long time, before the dynamics remove it towards the above expanding, accelerating, late-time attractors. The duration of these tran-

sient phases depends on the specific initial conditions.

One of the most interesting behaviors is the possibility of the realization of the transition between expanding and contracting solutions during the evolution. That is, the scenario at hand can exhibit the cosmological bounce or turnaround. Additionally, there can also appear an eternal transition between expanding and contracting phases, that is we can obtain cyclic cosmology. These features can be of great significance for cosmology, since they are desirable in order for a model to be free of past or future singularities.

Before closing, let us make some comments concerning the use of observational data in order to constrain the present scenario. In particular, equation (31), along with the R^n-ansatz, relation (34), and the definitions of the density parameters (23)-(26), can be straightforwardly written in the form used for observational fitting [59]. Thus, one can use observational data from Type Ia Supernovae (SNIa), Baryon Acoustic Oscillations (BAO), and Cosmic Microwave Background (CMB), along with requirements of Big Bang Nucleosynthesis (BBN), to constrain the model parameter n. Additionally, one can constrain the initial allowed anisotropy value σ_+ (or Ω_σ), through its present value $\Omega_{\sigma 0}$ and the shear evolution (30). Such a procedure is necessary for every cosmological paradigm, and can significantly enlighten the scenario at hand. However, since in the present work we focused on the dynamical features, the detailed observational elaboration is left for a separate project.

In summary, anisotropic R^n-gravity has a very rich cosmological behavior, and a large variety of evolutions and late-time solutions, compatible with observations. The much more complicated structure of anisotropic geometries leads to radically different implications comparing to the simple isotropic scenarios. These features indicate that anisotropic universes governed by modified gravity can be a candidate for the description of nature, and deserve further investigation.

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Appendix A: Eigenvalues of the perturbation matrix Ξ for the critical points

The system (46)-(49) admits ten isolated critical points presented in Table II. Here we provide the eigenvalues of the perturbation matrix Ξ calculated at each critical point. We also provide the eigenvalues of the perturbation matrix at the special points P_1^ϵ and P_2^ϵ located at the invariant circles C_ϵ.

For the two critical points $\mathcal{A}_\epsilon$ the associated eigenvalues read

$$\left\{-\epsilon\frac{2\left(9w^2+12w+1\right)}{3(3w+1)}, -\epsilon\frac{2(3w+2)(6w+1)}{3(3w+1)}[\times 2], -\epsilon\frac{(w+1)\left(9w^2+12w+1\right)}{3w+1}\right\},$$

with $[\times 2]$ denoting multiplicity 2.

For the critical points $\mathcal{B}_\epsilon$ the associated eigenvalues are

$$\left\{-\epsilon\frac{2(3w+1)}{3(w-1)}, -\epsilon\frac{(3w-2)(3w+1)}{3(w-1)}[\times 2], -\epsilon\frac{2(w+1)}{w-1}\right\}.$$

For the critical points P_1^ϵ the associated eigenvalues read

$$\left\{0, \epsilon(3w+1), 3\epsilon(3w+1), -\frac{3}{2}\epsilon(w-1)(3w+1)\right\},$$

while for P_2^ϵ they write

$$\left\{0, 3\epsilon(3w+1)[\times 2], -\frac{3}{2}(w-1)(3w+1)\right\}.$$

For the critical points P_3^ϵ the associated eigenvalues are

$$\{0, -\epsilon(3w+1), -\epsilon\frac{9w^2+\sqrt{3}\sqrt{3w+1}\sqrt{9w^3+21w^2+31w+3}-1}{2(3w-1)}, -\epsilon\frac{9w^2-\sqrt{3}\sqrt{3w+1}\sqrt{9w^3+21w^2+31w+3}-1}{2(3w-1)}\right\},$$

and thus there exists a 1-dimensional center manifold tangent to the line

$$\left\{x=-\frac{1}{2}Q(3w+1), y=\epsilon\frac{Q(w-1)(3w+1)^2}{(w+1)(3w-1)}, \Sigma=-\frac{1}{2}Q(3w-1),\, Q\in[-2,2]\right\}.$$

For the critical points P_4^ϵ the associated eigenvalues are

$$\left\{-\epsilon\frac{6(3w+1)\left(9w^2+3w+1\right)}{63w^2+30w+7},-\epsilon\frac{3(3w+1)\left(9w^2+3w+\sqrt{9w^2+3w+1}\sqrt{45w^2+51w+5}+1\right)}{63w^2+30w+7},\right.$$
$$\left.-\epsilon\frac{3(w+1)(3w+1)\left(9w^2+12w+1\right)}{63w^2+30w+7},-\epsilon\frac{3(3w+1)\left(9w^2+3w-\sqrt{9w^2+3w+1}\sqrt{45w^2+51w+5}+1\right)}{63w^2+30w+7}\right\}.$$

Finally, for the critical points P_5^ϵ the associated eigenvalues read

$$\left\{0,0,-\epsilon\frac{2(3w+1)\left(-3w+\sqrt{3w-2}+1\right)}{3w-1},6\epsilon(w+1)\right\}.$$

Appendix B: The direction ${E_1}^1$

In subsection III C we have investigated the system (46)-(49), leaving outside the decoupled equation (50). However, this equation provides information of how Kantowsky-Sachs $f(R)$ models are related to more general anisotropic geometries. For instance from (50) we see that ${E_1}^1=0$ is an invariant set of (46)-(50). This set is closely related with the so-called "silent boundary" one [60–62]. [4] Note that, as current investigations suggest, in G_0-cosmologies, that is in cosmologies where there is no symmetry, both past and future attractors belong to the silent boundary [61, 64].

In the present work we do not focus in how Kantowsky-Sachs $f(R)$ are related to more general anisotropic models, and thus, we do not elaborate completely equation (50). However, we can still obtain the corresponding significant physical information, since the stability along the ${E_1}^1$ direction is determined by calculating the sign of $\partial\left({E_1}^1\right)'/\partial {E_1}^1$, which coincides with the eigenvalue associated to the eigenvector ${E_1}^1$. Thus, if it is negative then the small perturbations in the ${E_1}^1$ direction decay, while if it is positive they enhance.

From the sign of $\partial\left({E_1}^1\right)'/\partial {E_1}^1|_{\mathcal{A}_\epsilon}=-\frac{\left(9w^2+12w+1\right)\epsilon}{3(3w+1)}$ it follows that $\mathcal{A}_+$ ($\mathcal{A}_-$) is stable (unstable) to small perturbations in the ${E_1}^1$ direction provided $w_+<w\leq 1$, otherwise it is unstable (stable).

From the sign of $\partial\left({E_1}^1\right)'/\partial {E_1}^1|_{\mathcal{B}_\epsilon}=-\frac{(3w+1)\epsilon}{3(w-1)}$ it follows that $\mathcal{B}_+$ ($\mathcal{B}_-$) is unstable (stable) to small perturbations in the ${E_1}^1$ direction.

From the signs of $\partial\left({E_1}^1\right)'/\partial {E_1}^1|_{P_1^\epsilon}=2\epsilon\,(3w+1)$ it follows that P_1^- (P_1^+) is stable (unstable) to small perturbations in the ${E_1}^1$ direction provided $-\frac{1}{3}<w<1$, otherwise it is unstable (stable). Since $\partial\left({E_1}^1\right)'/\partial {E_1}^1$ vanishes at P_2^ϵ and at P_3^ϵ, it follows that they are neutrally stable to small perturbation in the ${E_1}^1$ direction.

By analyzing the sign of $\partial\left({E_1}^1\right)'/\partial {E_1}^1|_{P_4^\epsilon}=-\frac{3(3w+1)\left(9w^2+12w+1\right)\epsilon}{63w^2+30w+7}$ we deduce that P_4^+ (P_4^-) is stable (unstable) to perturbations in the ${E_1}^1$ direction if $w_+<w\leq 1$.

From the sign of $\partial\left({E_1}^1\right)'/\partial {E_1}^1|_{P_5^\epsilon}=\frac{(3w+1)\left((3w-1)\epsilon+2\sqrt{3w-2}\right)}{3w-1}$, it follows that P_5^+ (P_5^-), whenever exists, is always unstable (stable) to small perturbations in the ${E_1}^1$ direction.

Finally, note that since

$$\frac{\partial\left({E_1}^1\right)'}{\partial {E_1}^1}|_{C_\epsilon}=2(n-1)[\epsilon+\cos(u)],$$

then for $1<n\leq 3$, C_+ is always unstable (except for $u=\pi$) to small perturbations in the ${E_1}^1$ direction, while C_- is always stable (except for $u\in\{0,2\pi\}$).

Lastly, it is interesting to mention that all the critical points of the reduced system, except $P_{2,3}^\epsilon$, belong to the "silent boundary" in the full phase space, that is each one has a representative in $\Psi\times\mathbb{R}$ with ${E_1}^1=0$. In particular, the critical points $\mathcal{A}_\epsilon$, $\mathcal{B}_\epsilon$ correspond to isotropic silent singularities of the full five-dimensional phase space and P_5^ϵ correspond to an anisotropic one.

[4] The concept of "silent boundary" was introduced for inhomogeneous cosmologies and it is defined by the condition $e_1^1/\beta=0$ where $\beta=H+\sigma_+$, with the β-gradient being equal to zero. It corresponds to the divergences of $H+\sigma_+$. In particular, as the initial singularity is approached, the radial part of the metric tends to zero and the null cone becomes a timeline (see section 5.3 of [63] for a clear physical discussion). In our analysis we do not use β-normalization, however, as $E_1^1=0$ is approached, a similar physical behavior is attained (that is the radial part of the metric tends to zero and the local null cone collapses onto the timeline).

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